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PROGRAMME: B.Sc. PHYSICS HONOURS

SEMESTER - VI

Course: Electromagnetic Theory (CC-XIII)

Lecture Note: 1

Maxwell's Equations and Gauge Transformation

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Maxwell's Equation

In electrodynamics, the Maxwell's equations in free space in SI unit are

i) $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$

ii) $\nabla\cdot\vec{B} = 0$

iii) $\vec{\nabla}\times\vec{E} = -\partial B/\partial t$.

iv) $\vec{\nabla}\times\vec{B} = \mu_0(J + \epsilon_0 \frac{\partial \vec{E}}{\partial t})$.

where ρ is the charge density.
J " " Current "
and $\epsilon_0 \frac{\partial \vec{E}}{\partial t}$ = displacement Current density.

These four equation describes an electromagnetic field through two vectors $\vec{E}$ & $\vec{B}$. In charge free space and with zero conductivity $\rho = 0$, and $\sigma = 0$, so $\vec{J} = \sigma\vec{E} = 0$.

So then we get the Maxwell's eq^ns

i) $\vec{\nabla}\cdot\vec{E} = 0$

ii) $\vec{\nabla}\cdot\vec{B} = 0$

iii) $\vec{\nabla}\times\vec{E} = -\partial B/\partial t$

iv) $\vec{\nabla}\times\vec{B} = \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t}$.

The Maxwell's eq^ns in matter

i) $\vec{\nabla}\cdot\vec{D} = \rho_f$, free charge density.

ii) $\vec{\nabla}\cdot\vec{B} = 0$

iii) $\vec{\nabla}\times\vec{E} = -\partial B/\partial t$

iv) $\vec{\nabla}\times\vec{H} = J_f + \frac{\partial \vec{D}}{\partial t}$.

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where, $\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon \vec{E}$ displacement vector.

ρ_f = free charge density

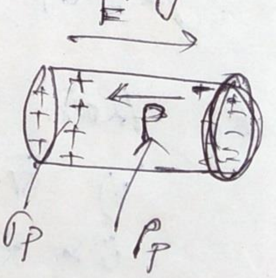
J_f = " Current density

$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$, electric field vector.

$= \vec{B}/\mu$.

In the presence of e.m field the material medium are electrically and magnetically polarised and inside the polarised matter there will be accumulation of bound charges and currents over which we don't have any direct control.

In electrostatics because of polarisation $\vec{P}$ there are accumulation of bound volume charge of density $\rho_P = -\vec{\nabla}\cdot\vec{P}$ and surface charge density $\sigma_P = \vec{P}\cdot\hat{n}$.



Due to magnetic polarization $\vec{M}$, there appear a bound current density, $\vec{J_m} = \vec{\nabla}\times\vec{M}$,

It involves the spin and orbital motion of electron.

In time varying field the change in electric polarization $\vec{P}$ involves a linear flow of bound

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charge. As a ~~result~~ result the Current density ~~Jp~~ due to polarization

$$J_p = \frac{\partial \vec{P}}{\partial t}.$$

The divergence of $\vec{J_p}$ is

$$\vec{\nabla}\cdot\vec{J_p} = \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{P}) = \frac{\partial \rho_p}{\partial t}.$$

The total charge density in matter is then

$$\rho = \rho_f + \rho_p = \rho_f - \vec{\nabla}\cdot\vec{P}.$$

The total current density —

$$\vec{J} = \vec{J_f} + \vec{J_p} + \vec{J_m}$$

Due to → free charge, electric polarization, magnetic polarization.

$$= \vec{J_f} + \frac{\partial \vec{P}}{\partial t} + \vec{\nabla}\times\vec{M}.$$

$$\vec{J} = \sigma\vec{E} + \frac{\partial \vec{P}}{\partial t} + \vec{\nabla}\times\vec{M}$$

The 1st Maxwell eqn is then

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$$

$$= \frac{\rho_f - \vec{\nabla}\cdot\vec{P}}{\epsilon_0}$$

or $\vec{\nabla}\cdot(\vec{E}\epsilon_0 + \vec{P}) = \rho_f$

or $\boxed{\vec{\nabla}\cdot\vec{D} = \rho_f},$ where $\vec{D} = \epsilon_0\vec{E} + \vec{P}$

From modified Ampere's law

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}$$

$$= \mu_0(\vec{J_f} + \vec{J_p} + \vec{J_m}) + \mu_0\epsilon_0\frac{\partial E}{\partial t}$$

$$= \mu_0\left(J_f + \frac{\partial \vec{P}}{\partial t} + \vec{\nabla}\times\vec{M}\right) + \mu_0\epsilon_0\frac{\partial E}{\partial t}$$

or $\nabla\times\left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = J_f + \frac{\partial}{\partial t}(\epsilon_0\vec{E} + \vec{P})$

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or $\boxed{\vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}}$

where, $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$, $\quad \vec{D} = \epsilon_0\vec{E} + \vec{P}$

or $\vec{B} = \mu\vec{H}$, $\quad = \epsilon\vec{E}$.

So the Maxwell's eq^n in matter are

i) $\vec{\nabla}\cdot\vec{D} = \rho_f$, iii) $\nabla\times\vec{E} = -\partial\vec{B}/\partial t$

ii) $\vec{\nabla}\cdot\vec{B} = 0$, iv) $\vec{\nabla}\times\vec{H} = J_f + \frac{\partial \vec{D}}{\partial t}$.

Potential formulation of Electrodynamics

The electromagnetic field is completely described by four Maxwell's equations. It is convenient to reduce the no. of equations by introducing two new ~~varib~~ variables ϕ, the scalar electric potential and $\vec{A}$, magnetic vector potential.

In electrostatics, $\vec{E}$ field is conservative i.e

$$\vec{\nabla}\times\vec{E} = 0$$

$$\text{or } \vec{E} = -\vec{\nabla}\phi$$

But in electrodynamics, this is no longer possible.

The magnetic field is solenoidal and

$$\vec{B} = \vec{\nabla}\times\vec{A} \quad \text{--- (1)}$$

Then from Maxwell's 3rd eq^n

$$\vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A})$$

$$= -\nabla\times\frac{\partial\vec{A}}{\partial t}$$

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$$\Rightarrow \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0 \quad \text{--- (2)}$$

or $$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla\phi,$$

as in electrodynamics $\vec{E} + \frac{\partial \vec{A}}{\partial t}$ is Conservative rather than $\vec{E}$ only.

so we get $$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \quad \text{--- (3)}$$

where ϕ is electric scalar potential.

So the two new variables ϕ and $\vec{A}$ represents the electric field vector and magnetic field vector through the eq^n (1) and (3).

The Maxwell's 1st eq^n is then

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$$

$$\Rightarrow \nabla\cdot\left(-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}\right) = \rho/\epsilon_0$$

$$\Rightarrow \nabla^2\phi + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\rho/\epsilon_0 \quad \text{--- (4)}$$

This reduces to the Poisson eq^n in Static case i.e

$$\nabla^2\phi = -\rho/\epsilon_0, \quad \left[\text{as then } \vec{A} = 0\right].$$

The forth Maxwell equation

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}$$

as $\vec{B} = \vec{\nabla}\times\vec{A}$, we get

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial}{\partial t}\left(-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}\right)$$

$$\Rightarrow \nabla(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J} + \mu_0\epsilon_0\left[\nabla\left(\frac{\partial \phi}{\partial t}\right) - \frac{\partial^2 \vec{A}}{\partial t^2}\right]$$

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$$\Rightarrow \nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} - \nabla\left[(\vec{\nabla}\cdot\vec{A}) + \mu_0\epsilon_0\frac{\partial\phi}{\partial t}\right] = -\mu_0\vec{J} \quad \text{--- (5)}$$

The eqn (4) and (5) carry all the information present in Maxwell's ~~forth~~ equation. This means the two potential ϕ and $\vec{A}$ represents the electromagnetic field through eqn (4) and (5). The eqn (4) and (5) are called ugly equation. These two ugly equation look a simple form by imposing a certain restriction known as Lorentz Gauge which is

$$\vec{\nabla}\cdot\vec{A} = -\mu_0\epsilon_0\frac{\partial\phi}{\partial t}.$$

So equation (4) and (5) are modified as

$$\nabla^2\phi - \mu_0\epsilon_0\frac{\partial^2\phi}{\partial t^2} = -\rho/\epsilon_0 \quad \text{--- (4.1)}$$

and

$$\nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}. \quad \text{--- (5.1)}$$

These two are the inhomogeneous wave equation. For static case (4.1) and (5.1) are ~~produce~~ reduce to

$$\nabla^2\phi = -\rho/\epsilon_0 \text{ (Poisson's eqn)}$$

and

$$\nabla^2\vec{A} = -\mu_0\vec{J} \text{ (Poisson's eqn in matter)}$$

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Gauge Transformation :-

From Maxwell 2nd equation

$$\vec{\nabla}\cdot\vec{B} = 0$$

$\therefore \quad \vec{B} = \vec{\nabla}\times\vec{A}$, where $\vec{A}$ is the magnetic vector potential. ——— (1)

So the relation (1) determined the magnetic field vector $\vec{B}$, by the specification of vector potential $\vec{A}$ of electromagnetic field. However it does not specified uniquely the vector potential $\vec{A}$. This is because a vector is not completely defined by stating its curl only, but by divergence also.

If we add to $\vec{A}$ vector the gradient of a scalar λ (say), the new vector potential is then

$$\vec{A}' = \vec{A} + \vec{\nabla}\lambda. \quad \text{——— (2)}$$

The magnetic field corresponds to new vector potential $\vec{A}'$ is

$$\vec{B}' = \vec{\nabla}\times\vec{A}'$$

$$= \nabla\times(\vec{A} + \vec{\nabla}\lambda)$$

$$= \vec{\nabla}\times\vec{A} + \nabla\times(\vec{\nabla}\lambda)$$

$$= \vec{\nabla}\times\vec{A} \quad [\because \nabla\times(\nabla\lambda) = 0]$$

or $\vec{B}' = \vec{B}$.

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This means with respect to transformation represented by eqn (2) the magnetic field vector, $\vec{B}$ remains uneffected or invarient.

When this is done the electric field vector

$$\vec{E}' = -\nabla\phi - \frac{\partial \vec{A}'}{\partial t}$$

$$= -\nabla\phi - \frac{\partial}{\partial t}(\vec{A} + \nabla\lambda)$$

$$= -\nabla\left(\phi + \frac{\partial\lambda}{\partial t}\right) - \frac{\partial\vec{A}}{\partial t} \quad (3)$$

$$\neq \vec{E}.$$

So with respect to transformation represented by eqn (2), the magnetic field vector $\vec{B}$ remains invarient but the electric field vector is changed. However if we transform $\phi \to \phi'$ such that

$$\phi' = \phi - \frac{\partial\lambda}{\partial t}, \quad (4)$$

then the electric field vector remains invarient.

These two transformation represented by equation (2) and (4) with respect to which the field vectors ($\vec{E}$ and $\vec{B}$) remains invarient, are known as Gauge transformation. The scalar function λ is known as Gauge function.

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It is the field quantities $\vec{E}$ & $\vec{B}$, ~~and~~ not the potentials (ϕ and $\vec{A}$) ~~that~~ possess physical meaningfullness. We then conclude the field vector are Gauge ~~transformation~~ invarient.

Because of arbitariness in the choice of Gauge function (Scalar λ) we are quite free to impose additional condition on a vector with a view to simplify the ugly equation.

In magnetostatics, $\nabla \cdot \vec{A} = 0$. is the simplest choice and it is known as Coulomb gauge. The other gauge which one ~~used~~ ~~the~~ use frequently in electrodynamics for a considerable simplification is $\vec{\nabla} \cdot \vec{A} + \epsilon_0 \mu_0 \frac{\partial \phi}{\partial t} = 0$.

$$\text{or} \quad \vec{\nabla} \cdot \vec{A} = -\epsilon_0 \mu_0 \frac{\partial \phi}{\partial t} \qquad (6)$$

This is known as ~~Lol.~~ Lorentz gauge of potential.

The scalar λ which was so far arbitary will now satisfy a condition

$$\vec{\nabla} \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial \phi}{\partial t} = 0$$

When $\vec{A} - \vec{A}'$ and $\phi - \phi'$, then this equation becomes

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$$\nabla\cdot(\bar{A}+\nabla\lambda)+\mu_0\epsilon_0\frac{\partial}{\partial t}\left(\phi-\frac{\partial\lambda}{\partial t}\right)=0$$

$$\Rightarrow \left(\nabla\cdot A+\mu_0\epsilon_0\frac{\partial\phi}{\partial t}\right)+\nabla^2\lambda-\mu_0\epsilon_0\frac{\partial^2\lambda}{\partial t^2}=0$$

$$\Rightarrow \boxed{\nabla^2\lambda-\mu_0\epsilon_0\frac{\partial^2\lambda}{\partial t^2}=0} \qquad \text{— (7)}$$

[As $\nabla\cdot A+\mu_0\epsilon_0\frac{\partial\phi}{\partial t}=0$ ~~from~~ which is Lorentz Gauge]

So if the potential $\vec{A}$, ϕ and $\vec{A}'$, ϕ' are to obey Lorentz condition, λ must ~~...~~ satisfy the wave equation represented by eqn (7).

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7.3.6 Boundary Conditions

In general, the fields **E, B, D,** and **H** will be discontinuous at a boundary between two different media, or at a surface that carries charge density σ or current density **K**. The explicit form of these discontinuities can be deduced from Maxwell's equations (7.55), in their integral form

$$\left.\begin{array}{ll}\text{(i)} & \oint_{\mathcal{S}} \mathbf{D}\cdot d\mathbf{a} = Q_{f_{\text{enc}}} \\ \text{(ii)} & \oint_{\mathcal{S}} \mathbf{B}\cdot d\mathbf{a} = 0\end{array}\right\} \text{ over any closed surface } \mathcal{S}.$$

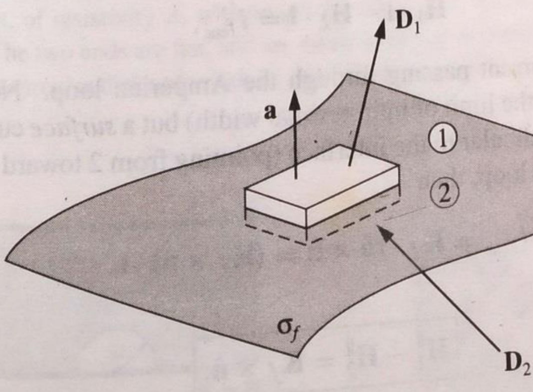
$$\left.\begin{array}{ll}\text{(iii)} & \oint_{\mathcal{P}} \mathbf{E}\cdot d\mathbf{l} = -\dfrac{d}{dt}\int_{\mathcal{S}} \mathbf{B}\cdot d\mathbf{a} \\ \text{(iv)} & \oint_{\mathcal{P}} \mathbf{H}\cdot d\mathbf{l} = I_{f_{\text{enc}}} + \dfrac{d}{dt}\int_{\mathcal{S}} \mathbf{D}\cdot d\mathbf{a}\end{array}\right\} \begin{array}{l}\text{for any surface } \mathcal{S} \\ \text{bounded by the} \\ \text{closed loop } \mathcal{P}.\end{array}$$

Applying (i) to a tiny, wafer-thin Gaussian pillbox extending just slightly into the material on either side of the boundary, we obtain (Fig. 7.46):

$$\mathbf{D}_1\cdot\mathbf{a} - \mathbf{D}_2\cdot\mathbf{a} = \sigma_f\, a.$$

(The positive direction for **a** is *from* 2 *toward* 1. The edge of the wafer contributes nothing in the limit as the thickness goes to zero, nor does any *volume* change density.) Thus, the component of **D** that is perpendicular to the interface is discontinuous in the amount

$$\boxed{D_1^{\perp} - D_2^{\perp} = \sigma_f.} \tag{7.59}$$



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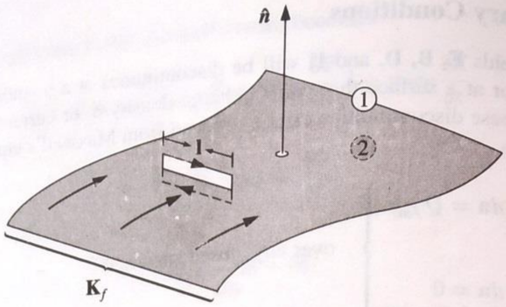

Figure 7.47

Identical reasoning, applied to equation (ii), yields

$$\boxed{B_1^{\perp} - B_2^{\perp} = 0.} \tag{7.60}$$

Turning to (iii), a very thin Amperian loop straddling the surface (Fig. 7.47) gives

$$\mathbf{E}_1 \cdot \mathbf{l} - \mathbf{E}_2 \cdot \mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{a}.$$

But in the limit as the width of the loop goes to zero, the flux vanishes. (I have already dropped the contribution of the two ends to $\oint \mathbf{E} \cdot d\mathbf{l}$, on the same grounds.) Therefore,

$$\boxed{\mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0.} \tag{7.61}$$

That is, the components of **E** *parallel* to the interface are continuous across the boundary. By the same token, (iv) implies

$$\mathbf{H}_1 \cdot \mathbf{l} - \mathbf{H}_2 \cdot \mathbf{l} = I_{f_{\text{enc}}},$$

where $I_{f_{\text{enc}}}$ is the free current passing through the Amperian loop. No *volume* current density will contribute (in the limit of infinitesimal width) but a *surface* current can. In fact, if $\hat{\mathbf{n}}$ is a unit vector perpendicular to the interface (pointing from 2 toward 1), so that $(\hat{\mathbf{n}} \times \mathbf{l})$ is normal to the Amperian loop, then

$$I_{f_{\text{enc}}} = \mathbf{K}_f \cdot (\hat{\mathbf{n}} \times \mathbf{l}) = (\mathbf{K}_f \times \hat{\mathbf{n}}) \cdot \mathbf{l},$$

and hence

$$\boxed{\mathbf{H}_1^{\parallel} - \mathbf{H}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}.} \tag{7.62}$$

So the *parallel* components of **H** are discontinuous by an amount proportional to the free surface current density.

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Equations 7.59-62 are the general boundary conditions for electrodynamics. In the case of *linear* media, they can be expressed in terms of **E** and **B** alone:

$$\left.\begin{array}{ll}\text{(i)}\ \epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f, & \text{(iii)}\ \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0, \\ \text{(ii)}\ B_1^{\perp} - B_2^{\perp} = 0, & \text{(iv)}\ \dfrac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \dfrac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}. \end{array}\right\} \tag{7.63}$$

In particular, if there is no free charge or free current at the interface, then

$$\boxed{\begin{array}{ll}\text{(i)}\ \epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = 0, & \text{(iii)}\ \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0, \\ \text{(ii)}\ B_1^{\perp} - B_2^{\perp} = 0, & \text{(iv)}\ \dfrac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \dfrac{1}{\mu_2}\mathbf{B}_2^{\parallel} = 0. \end{array}} \tag{7.64}$$

As we shall see in Chapter 9, these equations are the basis for the theory of reflection and refraction.

More Problems on Chapter 7

Problem 7.38 Two very large metal plates are held a distance d apart, one at potential zero, the other at potential V_0 (Fig. 7.48). A metal sphere of radius a ($a \ll d$) is sliced in two, and one hemisphere placed on the grounded plate, so that its potential is likewise zero. If the region between the plates is filled with weakly conducting material of uniform conductivity σ, what current flows to the hemisphere? [*Answer:* $(3\pi a^2\sigma/d)V_0$. *Hint:* study Ex. 3.8.]

! **Problem 7.39** Two long, straight copper pipes, each of radius a, are held a distance $2d$ apart (see Fig. 7.49). One is at potential V_0, the other at $-V_0$. The space surrounding the pipes is filled with weakly conducting material of conductivity σ. Find the current, per unit length, which flows from one pipe to the other. [*Hint:* refer to Prob. 3.11.]

Problem 7.40 A common textbook problem asks you to calculate the resistance of a cone-shaped object, of resistivity ρ, with length L, radius a at one end, and radius b at the other (Fig. 7.50). The two ends are flat, and are taken to be equipotentials. The suggested method is to slice it into circular disks of width dz, find the resistance of each disk, and integrate to get the total.

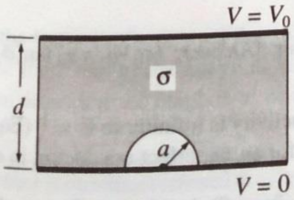

Figure 7.48

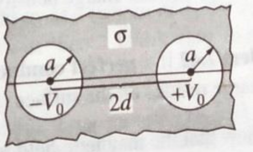

Figure 7.49